International Journal of Theoretical Physics, Vol. 11, No. 3 (1974), pp. 187–192

Symmetry in Quantum Systems

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Received: 20 *February* 1974

Abstract

Homological algebra is used to derive some results in the theory of Lie groups.

1. *Introduction*

The states of a quantum mechanical system form a projective Hilbert space. If the physical system admits a symmetry group, it induces automorphisms in the quantum space, i.e., a projective representation of the group. In this note we remark on this mathematical problem, using a minimum of modern homological algebra.

The problem as such was recognised early by Weyl (1950); Wigner (1939) studied the case for the Poincaré group, and Bargmann (1954) solved the situation for nearly a general Lie group, including the Galilei group; on the mathematical side Mackey (1958) initiated the work for any locally compact second-countable groups; for a self-contained exposition the book of Varadarajan (1968) should be consulted.

In Section 2 we set up the equivalence of the projective representation problem with a specific extension problem. Some simple results follow at once; the connexion of Lie group extension with Lie algebra extension is then used to obtain other results with Lie symmetry; in Section 3 the relation of the Galilei group with Heisenberg commutation rules is discussed.

As the paper is intended for physicists, modern but elementary algebraic techniques are used. We include an Appendix with the elements of homological algebra.

2. *The Extension Problem*

Let $\mathscr{H}$ be a separable complex Hilbert space and $\bar{\mathscr{H}}$ the associated projective space. The first theorem of Wigner (1959) asserts that

$$\mathrm{Aut}\,(\bar{\mathscr{H}}) = P\Gamma U(\mathscr{H}) \tag{2.1}$$

or in words: the group of transformations in $\bar{\mathscr{H}}$ which maintain the 'interference' of two rays **u**, **v** in $\bar{\mathscr{H}}$:

$$\mathbf{u} \,.\, \mathbf{v} = \frac{|\langle u | v \rangle|}{\|u\| \|v\|} \tag{2.2}$$

where u, v are arbitrary representative vectors in $\mathscr{H}$, is called Aut $(\bar{\mathscr{H}})$. ΓU is the group of unitary or antiunitary operators in $\mathscr{H}$, and $P\Gamma U$ is the image of ΓU in the projective group.

Wigner's theorem is just the fundamental theorem of projective geometry (e.g., Artin (1961)) extended to Hilbert spaces and restricted to the unitary part; a demonstration along the lines of projective geometry can be seen in Varadarajan, 1968, Vol. I, p. 167.

If $\mathscr{G}$ is the 'relativity' group of the physical system, there must be a morphism Δ: $\mathscr{G} \to$ Aut $(\bar{\mathscr{H}})$; in particular, the elementary quantum systems are given by the irreducible projective representations of $\mathscr{G}$, as first stated by Wigner (1939). As the projective group $P\Gamma U$ is $\Gamma U/U_1$, where U_1 = multiplicative group of unit modulus complex numbers, we resume the situation in the diagram with exact row

$$\begin{array}{ccccccccc} & & & & & & \mathscr{G} & & \\ & & & & & & \downarrow\Delta & & \\ 1 & \to & U_1 & \to & \Gamma U(\mathscr{H}) & \to & P\Gamma U(\mathscr{H}) & \to & 1 \end{array} \tag{2.3}$$

The relation of the problem of finding Δ (for a given $\mathscr{G}$) with some extensions of $\mathscr{G}$ (already recognised in the work of Bargmann (1954) if not earlier) stems from the following simple theorem (Maclane, 1963).

'The diagram (2.3) can be completed to the diagram

$$\begin{array}{ccccccccc} 1 & \to & U_1 & \to & H & \to & \mathscr{G} & \to & 1 \\ & & \| & & D\downarrow & & \downarrow\Delta & & \\ 1 & \to & U_1 & \to & \Gamma U & \to & P\Gamma U & \to & 1 \end{array} \tag{2.4}$$

which is commutative and with exact upper row; moreover $H(\mathscr{G}, U_1)$ is unique up to equivalence of extensions.'

Of course, the theorem is valid for any general situation $A \rightarrowtail B \overset{\pi}{\twoheadrightarrow} C$ and $\Delta : D \to C$; for the proof, one selects in $B \times D$ these pairs $H = \{b, d\}$ with $\pi(b) = \Delta(d)$; then $\bar{\pi}$: $(b, d) \to d$ is epic $H \twoheadrightarrow \mathscr{G}$, and ker $\bar{\pi} = A$; besides, $D(b, d) = b$ makes (2.4) commutative (for the algebraic concepts see Appendix).

As corollary, we obtain: if $1 \to A \to B \to C \to 1$ splits (i.e., if B is a semidirect product) then Δ prolongates to

$$\bar{D} = \sigma \circ \Delta : D \to B; \qquad \sigma : C \to B; \qquad \pi \circ \sigma = \text{Id } C \tag{2.5}$$

or: any morphism $D \to C$ 'lifts' to other $\bar{D} : D \to B$.

Thus for any relativity group $\mathscr{G}$ we have to compute the set Ext $(\mathscr{G}, U_1)$ of inequivalent extensions by U_1; in any extension $1 \to A \to B \to C \to 1$ the mapping $b : a \to bab^{-1} = a'$ gives, by quotient, a mapping $C \to \mathrm{Aut}\, A/\mathrm{Int}\, A \equiv \mathrm{Out}\, A$ (Michel, 1965); for A abelian, as in the physics case, $\mathrm{Aut}\, A = \mathrm{Out}\, A$, and one says that C operates in A; besides, in the extention (2.4) $\mathscr{G}$ operates in U_1 through $P\Gamma U$; but $g \in P\Gamma U$ operates in U, either trivially, $gz = z\,(z \in U_1)$ when g comes from a linear operator, or by conjugation $gz = z^{-1}$ when g is anti-linear. Therefore, only consider

$$\mathrm{Ext}_c\,(\mathscr{G}, U_1) \qquad c : \mathscr{G} \to Z_2 = \mathrm{Aut}\, U_1 \tag{2.6}$$

In particular, if $\mathscr{G}$ is a *connected* topological group and we restrict ourselves to continuous representations, $\Delta(\mathscr{G})$ is in the connected piece of $P\Gamma U$ and the extensions are *central*, i.e., $\mathscr{G} \to \mathrm{Id}_{\mathrm{Aut}\, U_1}$.† In the physical case if $\mathscr{G}$ is a topological group and $\mathscr{G}_0$ the (invariant) connected subgroup then $\mathscr{G}/\mathscr{G}_0$ is usually a finite group; the problem for $\mathscr{G}$ is the same as for $\mathscr{G}_0$, modulus some simple algebraic constructions.

Therefore taking $\mathscr{G}_0$ connected, the set‡

$$\mathrm{Ext}_0\,(\mathscr{G}_\theta, U_1) \tag{2.7}$$

of central extensions gives the classes of projective representations; for any extension one has then to calculate the unitary representations in Hilbert space, in particular the irreducible ones which imply, as is easily seen, irreducibility of the projective representation. The case of finite-dimensional representations is disposed of immediately, as the following Lemma states:

'Any finite-dimensional projective representation of a connected simply connected Lie group is induced by a linear representation.'

This is well known (Weyl, 1950; Bargmann, 1954); we offer a proof: if n is the dimension, then we have the identity

$$U_n = \frac{U_1 \times SU_n}{Z_n} \tag{2.8}$$

where Z_n is the cyclic group of nth roots of 1; (2.8) is obvious. As in the neighbourhood of 1 $U_n \approx U_1 \times SU_n$, locally $1 \to U_1 \to U_n \to SU_n \to 1$ is *split* (direct product), and any (close to 1) map $\mathscr{G} \to SU_n$ prolongates to $\mathscr{G} \to U_n$; for a simply connected Lie group the globalisation is unique, and the result holds.

If $\mathscr{G}_0$ is connected and $\tilde{\mathscr{G}}_0 \xrightarrow{\pi} \mathscr{G}_0$ is the universal covering, the representation problem for $\tilde{\mathscr{G}}_0$ includes that of $\mathscr{G}_0$: one has only to keep those $\tilde{\mathscr{G}}_0$-representations which map $\mathrm{Ker}\, \pi = \pi^1(\mathscr{G}_0)$ (which is a central discrete subgroup of $\tilde{\mathscr{G}}_0$) into the U_1 subgroups of U; by Schur's Lemma, this is true in particular for any (finite-dimensional) *irreducible* representation. The existence of the map: $U_n \to U_1$ given by the determinant does not exist for Hilbert spaces, and in fact the row of (2.3) is not split.

† If $\mathscr{G}_0$ is a connected *Lie* group, centrality of the extensions comes out algebraically, i.e., any $g \in G_c$ is a 'product of squares' $g = g_1^2 g_2^2 \ldots g_r^2$, and the square of an anti-unitary operator is unitary. See Wigner (1959).

‡ This set is the second cohomology group of $\mathscr{G}$ with values in U_1; see Maclane (1963), p. 112, or Michel (1965).

In the Hilbert space case the universal covering group of a Lie group is also important, due to 1-1 correspondence with analogous problems with *Lie algebras*; in fact, this is a very important part of Bargmann's (1954) work. We take his results without comment; the Lie algebra problem is then reduced to an easy computational problem in linear algebra; for abelian or semisimple Lie algebras Bargmann solves the problem explicitly.

We just want to proceed in the case in which $\mathscr{G}_0$ is such a Lie group so $\tilde{\mathscr{G}}_0$ is a semisimple Lie group; then Bargmann shows that any extension by U_1 splits; the projective problem for the $\mathscr{G}_0$ group can be visualised in the diagram

$$\begin{array}{c} 1 \to \pi^1 \to \tilde{\mathscr{G}}_0 \to \mathscr{G}_0 \to 1 \\ \downarrow \\ 1 \to U_1 \to U \to PU \to 1 \end{array} \tag{2.9}$$

so any morphism $\pi^1 \to U_1$, permits the induction of a projective representation $\mathscr{G}_0 \to PU$ by quotient; this corresponds to

$$\mathrm{Ext}_0\,(\mathscr{G}_0, U_1) = \mathrm{Hom}\,(\pi^1(\mathscr{G}_0), U_1) \tag{2.10}$$

already used by Michel (1964) for the Lorentz and Poincaré groups (in both cases, $\pi^1 = Z_2$, and $\mathrm{Hom}\,(Z_2, U_1) \equiv \hat{Z}_2$ = character group of $Z_2 = Z_2$ itself; the Lorentz group is simple, and its covering is also simple as a Lie group; the covering group of the Poincaré group is not semisimple, but has also only trivial central U_1 extensions; see Bargmann (1954).

As another application of (2.10), let us consider the case of the conformal group in Minkowski space, $\Gamma \approx O(4, 2)$; this is a simple Lie group (a real non-compact form of the $A_3 = D_3$ simple Lie algebra in Cartan classification); the maximal compact subgroup is $\approx SO_4 \times SO_2$, and hence the homotopy is $\approx Z$ (the covering group has infinite denumerable sheets). As $\hat{Z} = U_1$,

$$\mathrm{Ext}\,(\Gamma, U_1) = U_1 \tag{2.11}$$

and there exists a nondenumerable set of classes of projective unitary representations (these do not seem to have too much to do with physics; see Kastrup (1962) for the reasons why nonunitary representations are the pertinent ones).

3. *Galilei Group and Quantum Mechanics*

The Galilei group has nontrivial U_1 cohomology, i.e., there are nontrivial central extensions of the universal covering group of the Galilei group, as determined by Bargmann (1954). The difficulty already appears in the subgroup of translations and specific Galilei transformations in the same direction, which is isomorphic to the translation group in the plane, R^2; in fact, $\mathrm{Ext}_0\,(R^2, U_1) = R^1$ is immediate from the abelian and simply-connected nature of R^2, and also Ext_0 (Galilei, U_1) $= R^1$ (the parameter to fix $m \in R^1$ gives the mass of nonrelativistic systems). Any extensions of the R^2 group

$$1 \to U_1 \to H_m \to R^2 \to 1 \qquad m \in R^1 \tag{3.1}$$

is formed with the following composition law in $U_1 \times R^2$:

$$(Z; p, q)(Z'; p', q') = (Z + Z' + m(pq' - qp'); p + p', q + q') \qquad (3.2)$$

where $Z \in R^1$, $e^{iz} \in U_1$. If one passes to the Lie algebra, and i, P and Q are the generators of U_1, R^2, one has

$$[Q, P] = im \qquad (3.3)$$

which are precisely Heisenberg commutation rules for $m = \hbar$; this relation between the Heisenberg algebra and projective representations of the R^2 group was stressed in 1.927 by Weyl (1950); the relation with the Galilei group comes from Bargmann (1954). As von Neumann (1931) proved that (3.3) has only a (class of) faithful irreducible unitary representation (and, as a consequence, Heisenberg matrix mechanics and Schrödinger wave mechanics are equivalent), one has essentially only one (Heisenberg) group, but a R^1 manifolds of extensions (3.1). We want to show how this paradox is solved: any two groups $H_m, H_{m'}$ for $0 \neq m \neq m' \neq 0$ are isomorphism as abstract groups, but disequivalent as extensions.

That this is so comes from the fact that U_1 is a *divisible* group, i.e., $y = nx$ for $y, x \in U_1, n \in Z$ has always the solution; therefore the mapping $m : x \to mx$ is an *isomorphism* between U_1 and a quotient group; this prolongates to an isomorphism map $H_m \to H_{m'}$, but as the restriction of this to U_1 is obviously not the identity, H_m are disequivalent to $H_{m'}$, as it must be.

We conclude that the commutation rules fit very naturally with the Galilei group, whereas this is not so with the Poincaré group. In particular, the position operator has an invariant meaning only in non-relativistic quantum mechanics, namely as a multiple of the generator of specific Galilei transformations. Whilst this can be taken as a clue that Hilbert spaces are not the right frame for relativistic theories we leave the reader to judge.

Appendix

A morphism $\sigma = G \to G'$ is *epic* if $\sigma(G) = G'$, *monic* if Ker $\sigma = e$ (this is the most modern terminology used, e.g. in Ratman, 1970). If G_0 is an invariant subgroup of G, and $G/G_0 = Q$, we write equivalently $1 \to G_0 \overset{n}{\to} G \overset{\pi}{\to} Q \to 1$ as an *exact* sequence, i.e., the Kernel of a map is the image of the previous one; here n monic and π epic mean exactness in G_0 and Q. A set of groups and morphisms (arrows) is a *diagram*, in general; is commutative if the image of a point (through existing arrows) does not depend on the path.

An *extension* of the group C by the group A is an exact sequence $E = 1 \to A \to B \to C \to 1$, i.e., $B/A = C$, but remember that the extension is the set $E = (B, A, C)$; two E, E' are equivalent if there exist a morphism $f: E \to E'$ such that its restriction to A is the identity map, and so also is the quotient $\bar{f} = Q \to Q$; it follows that f^{-1} exists, and in fact f is an equivalence relation.

Any extension $E = (B; A, C)$ is determined by two things: first, a morphism $\sigma : C \to \mathrm{Out}\, A$, and a 'factor system' $\omega : C \times C \to A$ whose complete description is a bit long; if $\sigma(C) = 1$, the extension is called central; for some $\sigma \in \mathrm{Hom}\,(C,$

Out A) there are extensions; if A is abelian, Out A = Aut A, there are always extensions for any σ, and they make up an abelian group, written $H_\sigma^2(G, A)$; the unity is the *semidirect* extension (for a σ given), with composition law $(a, c) \circledcirc (a'c) = (a + \sigma_c(a'), cc')$. For abelian groups an extension central and semidirect is the direct product or trivial extension, and is unique.

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